g.f. application in SRW: $S_0 = 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad X_i = \begin{cases} 1 & \text{w.p. } p \\ -1 & \text{w.p. } q = 1-p \end{cases}$$

$$P_0(s) \triangleq \sum_{n=0}^{\infty} \underbrace{p_0(n)}_{\mathbb{P}(S_n=0)} \cdot s^n, \quad F_0(s) \triangleq \sum_{n=0}^{\infty} \underbrace{f_0(n)}_{\mathbb{P}(S_1 \neq 0, \cdots, S_{n-1} \neq 0, S_n = 0)} \cdot s^n$$

first hitting to zero happens at time n

then: $\begin{cases} P_0(s) = (1 - 4pqs^2)^{-\frac{1}{2}} \\ F_0(s) = 1 - (1 - 4pq s^2)^{\frac{1}{2}} \end{cases}$

similarly, $F_r(s) \triangleq \sum_{n=0}^{\infty} \underbrace{f_r(n)}_{\mathbb{P}(S_1 \neq r, \cdots, S_{n-1} \neq r, S_n = r)} \cdot s^n$

then $\underbrace{F_r(s) = [F_1(s)]^r}_{\text{Markov property}} = \dfrac{1 - (1 - 4pqs^2)^{\frac{1}{2}}}{2qs}$

eg: (5.3.2) For SSRW, show

(a): $2k f_0(2k) = \mathbb{P}(S_{2k-2}=0)$ for $k \geq 1$

Pf: $F_0(s) = \sum_{n=0}^{\infty} f_0(n)\cdot s^n = \sum_{k=0}^{\infty} f_0(2k)\cdot s^{2k} = 1-(1-4pqs^2)^{\frac{1}{2}}$

($f_0(n)=0$ if n is odd)

diff w.r.t. s, interchange diff and summation:

$$\sum_{k=1}^{\infty} 2k\cdot f_0(2k)\, s^{2k-1} = \frac{4pqs}{\sqrt{1-4pqs^2}}$$

$$\| \text{ Taylor}$$

$$4pqs\cdot \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}\cdot(-4pqs^2)^n$$

$$\|$$

$$4pqs\cdot \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n\cdot n!}(4pq)^n\cdot s^{2n}$$

$$\|$$

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n\cdot n!}(4pq)^{n+1} s^{2n+1}$$

identify $2k-1$ as $2n+1$: ($p=q=\frac{1}{2}$)

$$2kf_0(2k) = \frac{(2k-3)!!}{2^{k-1}\cdot(k-1)!} = \frac{2^{k-1}\cdot\frac{(2k-3)!!}{(k-1)!}}{2^{2k-2}} = \frac{\frac{(2k-2)!}{[(k-1)!]^2}}{2^{2k-2}}$$

$$= \frac{\binom{2k-2}{k-1}}{2^{2k-2}} = \mathbb{P}(S_{2k-2}=0) \text{ for } \forall k\geq 1.$$

(b): $\mathbb{P}(S_1 S_2 \cdots S_{2n} \neq 0) = \mathbb{P}(S_{2n} = 0)$ for $n \geq 1$

Pf: connect with first hitting time to 0, notice that it's possible to hit 0 only at even time

$$\text{LHS} = \mathbb{P}(S_1 \cdots S_{2n} \neq 0,\ S_{2n+2} = 0) + \mathbb{P}(S_1 \cdots S_{2n} S_{2n+1} S_{2n+2} \neq 0)$$

$$= f_0(2n+2) + \mathbb{P}(S_1 \cdots S_{2n} S_{2n+1} S_{2n+2} \neq 0)$$

$$= \cdots$$

$$= \sum_{\substack{k=2n+2 \\ k \text{ even}}}^{\infty} f_0(k) = \sum_{k=n+1}^{\infty} f_0(2k)$$

Only need to prove: $\displaystyle\sum_{n=0}^{\infty} \mathbb{P}(S_1 S_2 \cdots S_{2n} \neq 0) \cdot s^{2n}$

$$= \sum_{n=0}^{\infty} \mathbb{P}(S_{2n} = 0) \cdot s^{2n} = P_0(s)$$

$$= (1 - 4pqs^2)^{-\frac{1}{2}}$$

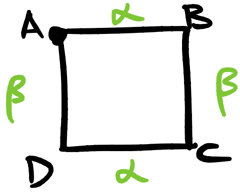
Calculate $\displaystyle\sum_{n=0}^{\infty} \mathbb{P}(S_1 S_2 \cdots S_{2n} \neq 0) \cdot s^{2n}$

$$= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} f_0(2k) s^{2n} \overset{\text{Fubini}}{=\!=} \sum_{k=1}^{\infty} f_0(2k) \sum_{n=0}^{k-1} s^{2n}$$

$$= \sum_{k=1}^{\infty} f_0(2k) \cdot \frac{1 - s^{2k}}{1 - s^2} \ \overset{\text{def of}}{\underset{\text{g.f.}}{=\!=}}\ \frac{1 - F_0(s)}{1 - s^2}$$

$$= \frac{\sqrt{1 - s^2}}{1 - s^2} = \frac{1}{\sqrt{1 - s^2}} = P_0(s) \quad \text{when } p = q = \tfrac{1}{2}.$$

e.g. (5.3.3)



$\alpha+\beta=1$ prob of moving

$G_A(s)$ is g.f. of $\{P_{AA}(n): n\geq 0\}$, prob of particle starting A is at A after n steps, calculate g.f. and find g.f. of first returning time to A.

Pf: $X_n \triangleq$ loc of particle at time n, $X_0=A$

$$P_{AA}(2n)=\underbrace{\mathbb{P}(X_{2n}=A)}_{\text{non-trivial only when } n \text{ is even}}=\mathbb{P}\left(\bigcup_{k=0}^{n}\left\{\begin{array}{c}2k \text{ times run through } AB \text{ and } DC,\\ 2(n-k) \text{ times run through } AD \text{ and } BC\end{array}\right\}\right)$$

$$=\sum_{k=0}^{n}\binom{2n}{2k}\alpha^{2k}\cdot\beta^{2(n-k)}=\beta^{2n}\cdot\sum_{k=0}^{n}\binom{2n}{2k}\left(\frac{\alpha}{\beta}\right)^{2k}$$

$$\begin{cases}\sum_{k=0}^{n}\binom{2n}{2k}\left(\frac{\alpha}{\beta}\right)^{2k}+\sum_{k=0}^{n-1}\binom{2n}{2k+1}\left(\frac{\alpha}{\beta}\right)^{2k+1}=\left(1+\frac{\alpha}{\beta}\right)^{2n}\\ \sum_{k=0}^{n}\binom{2n}{2k}\left(\frac{\alpha}{\beta}\right)^{2k}-\sum_{k=0}^{n-1}\binom{2n}{2k+1}\left(\frac{\alpha}{\beta}\right)^{2k+1}=\left(1-\frac{\alpha}{\beta}\right)^{2n}\end{cases}$$

$$=\beta^{2n}\cdot\frac{1}{2}\cdot\left[\left(1+\frac{\alpha}{\beta}\right)^{2n}+\left(1-\frac{\alpha}{\beta}\right)^{2n}\right]$$

$$=\frac{1}{2}\left[(\alpha+\beta)^{2n}+(\beta-\alpha)^{2n}\right]=\frac{1}{2}\left[1+(\beta-\alpha)^{2n}\right]$$

So: $G_A(s) = \sum_{n=0}^{\infty} P_{AA}(2n)\, s^{2n}$

$$= \frac{1}{2}\cdot\left(\sum_{n=0}^{\infty} s^{2n} + \sum_{n=0}^{\infty} \left[(\beta-\alpha)s\right]^{2n}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-s^2} + \frac{1}{1-\left[(\beta-\alpha)s\right]^2}\right)$$

Denote T as first returning time to A, and $f_A(n) = \mathbb{P}(T=n)$, we consider $F_A(s) \triangleq \sum_{n=0}^{\infty} f_A(n)\cdot s^n$

By first-hitting-time decomposition,

$$\mathbb{P}(X_{2n}=A) = \sum_{k=1}^{n} \mathbb{P}(T=2k)\cdot \underbrace{\mathbb{P}(X_{2n}=A \mid T=2k)}_{\Downarrow \text{ Markov property}}$$

$$= \sum_{k=1}^{n} \mathbb{P}(T=2k)\cdot \underline{\mathbb{P}(X_{2n-2k}=A)}$$

So $P_{AA}(2n) = \sum_{k=1}^{n} f_A(2k)\cdot P_{AA}(2n-2k)$ for $\forall n \geq 1$

multiply by s^{2n} and sum w.r.t. n to get:

$$G_A(s) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_A(2k)s^{2k}\cdot P_{AA}(2n-2k)\cdot s^{2n-2k}$$

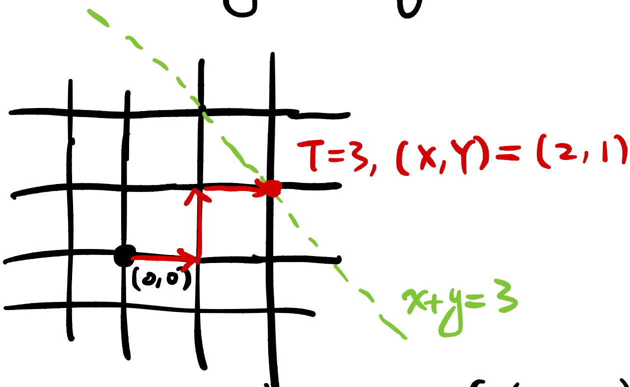
$$= 1 + \sum_{n=0}^{\infty} f_A(2n)s^{2n}\cdot \sum_{n=0}^{\infty} P_{AA}(2n)\cdot s^{2n}$$

$$= 1 + F_A(s)\cdot G_A(s)$$

So $$F_A(s) = \frac{G_A(s) - 1}{G_A(s)}.$$

e.g. (5.2.4) particle, SSRW on $\mathbb{Z}^2$, start at origin, each step length 1, with equal prob $\frac{1}{4}$ taking each direction. The particle reaches line $x+y=m$ at point (X,Y) at time T (first hitting time). Find g.f. of T and $X-Y$, specify convergence domain.

Pf:



At time n, has location (X_n, Y_n) so that

$$(X_{n+1}, Y_{n+1}) = (X_n, Y_n) + \begin{cases} (0,1) \\ (1,0) \\ (0,-1) \\ (-1,0) \end{cases} \quad \text{w.p. } \frac{1}{4} \quad (\text{indep movement})$$

check: $U_n \triangleq X_n + Y_n$, $U_{n+1} = U_n + \begin{cases} 1 \\ -1 \end{cases}$ w.p. $\frac{1}{2}$ (indep movement)

so U_n is SSRW on $\mathbb{Z}$.

$$V_n \triangleq X_n - Y_n \text{ is also SSRW on } \mathbb{Z}.$$

$T = \inf\{n : U_n = m\}$ is first hitting time to m,

$$G_T(s) = F_m(s) = \boxed{\left(\frac{1-\sqrt{1-s^2}}{s}\right)^m}$$

$X - Y = X_T - Y_T = V_T$ so

$G_{X-Y}(s) = \mathbb{E} s^{V_T} = \mathbb{E}\left[\mathbb{E}(s^{V_T} | T)\right]$

$$\mathbb{E}(s^{V_T} | T=t) = \mathbb{E}(s^{V_t} | T=t) = \mathbb{E} s^{V_t} = G_{V_t}(s)$$

due to independence of U and V.

To see this:

increment of (X_n, Y_n) as $\begin{cases} (-1,0) \\ (1,0) \\ (0,-1) \\ (0,1) \end{cases} \Longrightarrow$

ξ_n^U (increment of U_n)	ξ_n^V (increment of V_n)
-1	-1
1	1
-1	1
1	-1

those increments are independent!

e.g.:

$$\mathbb{P}(\xi_n^U = 1, \xi_n^V = 1) = \frac{1}{4} = \mathbb{P}(\xi_n^U = 1) \cdot \mathbb{P}(\xi_n^V = 1)$$

Also, $V_t = \sum_{k=1}^{t} \underbrace{\xi_k^V}_{\text{i.i.d.}}$, by property of g.f.,

$$G_{V_t}(s) = \left[G_{\xi_1^V}(s)\right]^t = \left(\frac{1}{2}s + \frac{1}{2}\frac{1}{s}\right)^t$$

$$= \mathbb{E}\left(\frac{s}{2} + \frac{1}{2s}\right)^T = \boxed{G_T\left(\frac{s}{2} + \frac{1}{2s}\right)}$$

Convergence Domain

$G_T(s) = \left(\frac{1-\sqrt{1-s^2}}{s}\right)^m$ well-defined for $s \in [-1, 1]$,

so for $G_{X-Y}(s)$, it's well-defined iff

$$\frac{s}{2} + \frac{1}{2s} \in [-1, 1] \Rightarrow s \in \{-1, 1\}$$